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LETTER TO THE EDITOR

Deterministic model for left-sided multifractality in generalized diffusion-limited aggregation

Takashi Nagatani

College of Engineering, Shizuoka University, Hamamatsu 432, Japan, and Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA

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Abstract. An exactly solvable deterministic model on a hierarchical lattice is presented for the left-sided multifractality of the growth probability distribution in a generalized diffusion-limited aggregation (η model). The exactly renormalizable growth probability is given by a multiplicative process. It predicts the tip behaviour (the maximum growth probability is a power law) and the fjord behaviour (the minimum growth probability has a logarithmic singularity). An analytical form of a 'left-sided' generalized dimension $D(q)$ is shown for any η. For the η model, the minimum growth probability scales as $p_{\min} \approx \exp[-c_1 \ln L - c_2(\ln L)^2]$. In the limit of $\eta \to 0$, $p_{\min} \to L^{-1}$.

The essential properties of kinetic aggregation processes are fully described by the growth probability distribution on the perimeter sites (or bonds) of these aggregation clusters. The growth probability can be regarded as a measure associated with each site (bond). The harmonic measure affords a method of quantitatively characterizing the relevant properties of the surfaces of the diffusion-limited aggregation clusters. A hierarchy of generalized dimensions $D(q)$ is used to characterize the harmonic measure (Halsey *et al* 1986, Amitrano *et al* 1986, Meakin 1988, Stanley and Ostrowsky 1988, Feder 1988, Vicsek 1989, Pietronero 1990).

Very recently, there has been much discussion of measures for which the partition function diverges faster than a power law, for small enough negative q values (Blumenfeld and Aharony 1989, Harris and Cohen 1990, Schwarzer *et al* 1990, Lee *et al* 1990, Mandelbrot *et al* 1990). There exist three recently proposed forms for the dependence on M of $p_{\min}$, the smallest of all the growth probabilities.

(I) Blumenfeld and Aharony (1989) and Mandelbrot *et al* (1990) proposed that $p_{\min}$ decreases exponentially with cluster mass M,

$$p_{\min}(M) \approx \exp(-cM^x). \qquad (1a)$$

(II) Mandelbrot and Vicsek (1989) and Harris and Cohen (1990) proposed the power-law form

$$p_{\min}(M) \approx M^{-\alpha_{\max}/d_f} \qquad (1b)$$

where d_f is the fractal dimension of DLA.

(III) Schwarzer *et al* (1990) carry out simulations for $p_{\min}$ of a typical DLA cluster. They find a surprising result for how $p_{\min}$ depends on cluster mass,

$$p_{\min}(M) \approx \exp[-c(\ln M)^y] \qquad (1c)$$

where $y \approx 2$.

Schwarzer *et al* (1990) and Lee *et al* (1990) construct a simple model for the fjord structure in DLA which predicts the form (1*c*) with $y=2$. Their model is based on the two assumptions of the hierarchical void-channel structure of the fjord. See the review paper by Stanley *et al* (1990) for details of the minimum growth probability and the references therein.

The 'free energy' $\tau(q)$ is singular at $q=0$ and fails to be defined for $q<0$ because of faster decreasing minimum growth probabilities than a power law. Mandelbrot *et al* called the 'anomalous' multifractal measures 'left-sided' multifractality. They introduce and investigate a family of exactly self-similar non-random fractal measures, each having stretched exponentially decreasing minimum probabilities.

In this letter, we present an exactly solvable model for the growth probability distribution on the surface of deterministic fractal aggregates on a hierarchical lattice. We address the dependence of the minimum growth probability distribution upon the parameter η. We show that the model has a 'left-sided' multifractality and the minimum growth probability $p_{\min}$ scales as

$$p_{\min} \approx \exp[-c_1 \ln L - c_2(\ln L)^2] \tag{2}$$

with

$$c_1 = [\ln 2 + \ln\{(5/3)^\eta + (1/6)^\eta\} + \eta \ln 3 + (\eta \ln 2)/2]/\ln 4$$

$$c_2 = \eta/(8 \ln 2)$$

where L is the cluster size. In the limit $\eta \to 0$, the minimum growth probability $p_{\min}$ scales as $p_{\min} \to L^{-1}$. It is shown that the minimum growth probability crosses over from $p_{\min} \approx \exp[-c(\ln L)^2]$ to $p_{\min} \approx L^{-1}$ with decreasing η. We obtain the analytical form of the 'left-sided' multifractality for any η.

We extend the previous model (Nagatani 1987) to take into account the fjord structure. The previous model can predict the tip behaviour (the moment of the growth probability has a power law). However, it cannot predict the fjord behaviour (the minimum growth probability decreases faster than a power law). We introduce a generator for the fjord structure into the previous model. Our model has no assumption for the scaling of the fjord. Only if one mimics the typical fjord structure by a deterministic fractal, can one exactly derive the left-sided multifractality of the growth probability distribution. Let us construct the deterministic fractal on a hierarchical lattice to mimic a typical DLA cluster. In general, the aggregates grown on lattices are viewed as a system of superconductor-normal resistor networks for the Laplacian growth model. The growth occurs on the perimeter of the aggregate. In the models the growth probability p_i at the growing perimeter bond i is given by $p_i \approx (I_i)^\eta$ where I_i is the local current at the growth bond i. Our deterministic fractal model is constructed by the four types of generators shown in figure 1(*b*), (*c*), (*d*) and (*e*) indicating, respectively, the generators for the superconducting bonds, the normal resistor bonds, the fjord bonds, and the tip bonds. Figure 1(*a*) shows the initiator. In the figure, the superconducting bond, the normal resistor bond, the fjord bond and the tip bond are respectively indicated by the thick line, the thin line, the wavy line, and the double wavy line. The method of constructing the deterministic fractal on a hierarchical lattice proceeds iteratively. The first generation is obtained from the zeroth generation (the initiator) by replacing each bond with each generator. The length scale is transformed by the factor $L_1=4$. The second generation is obtained from the first generation by replacing each bond with each generator. The resultant is scaled up to four times. The process is continued *ad infinitum*. In this way one can obtain the deterministic fractal

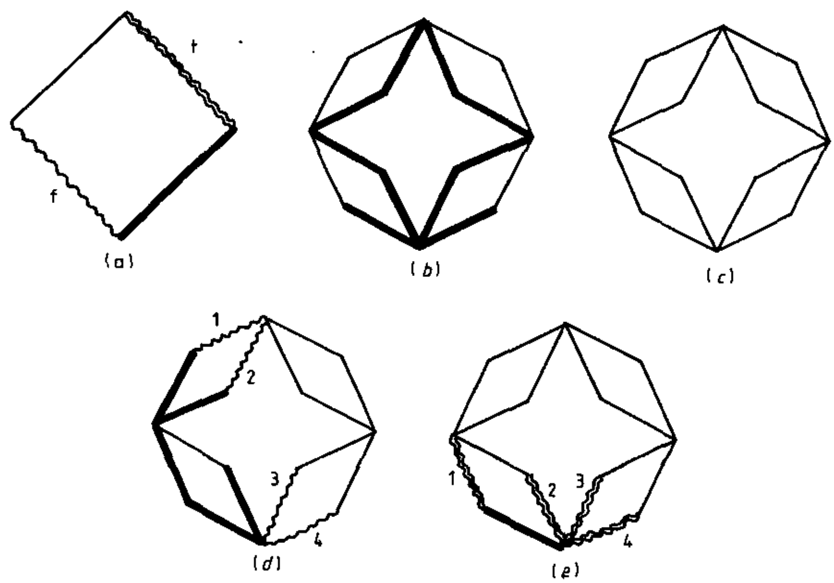

Figure 1. Initiator and generators for the deterministic fractal model on the hierarchical lattice. The superconducting, normal resistor, fjord and tip bonds are respectively indicated by the bold, light, wavy and double wavy lines. (*a*) The initiator. (*b*) The generator for the superconducting bonds. (*c*) The generator for the normal resistor bonds. (*d*) The generator for the fjord. (*e*) The generator for the tip.

aggregate on the hierarchical lattice. We note that the number S $(4 \leqslant S \leqslant 16)$ of superconductors within the generator for superconducting bonds (shown in figure 1(*b*)) is an adjustable parameter which is self-consistently determined by $D(\infty)$. The growth probability within the generator is determined by both cells' configurations and the conductances of the fjord and tip bonds. The growth probability on the deterministic fractal is given by the multiplicative process of each generator's growth probability because of the hierarchical structure. This deterministic fractal model is exactly renormalizable.

Consider the conductances between the top and the bottom at the nth generations of the tip and the fjord. The conductances of the tip bonds between the $(n-1)$th and the nth generations are related by the recursion relation

$$\sigma_{t,n-1} = (\sigma_{t,n}^2 + 2\sigma_{t,n})/(\sigma_{t,n}^2 + 3\sigma_{t,n} + 1) + 2\sigma_{t,n}/(3\sigma_{t,n} + 1) \tag{3}$$

where $\sigma_{t,n-1}$ and $\sigma_{t,n}$ are the conductances of the tip bonds at the $(n-1)$th and nth generations. Similarly, the conductances of the fjord bonds at the $(n-1)$th and nth generations are related by

$$\sigma_{f,n-1} = 2\sigma_{f,n} + 2\sigma_{f,n}/(3\sigma_{f,n} + 1) \tag{4}$$

where $\sigma_{f,n-1}$ and $\sigma_{f,n}$ are the conductances of the fjord bonds at the $(n-1)$th and nth generations. The growth probabilities within the generator for the tip at the nth generation are given by

$$\begin{aligned} p_{t,n,1} &= i_{n,1}^{\eta}/(i_{n,1}^{\eta} + i_{n,2}^{\eta} + 2i_{n,3}^{\eta}) \\ p_{t,n,2} &= i_{n,2}^{\eta}/(i_{n,1}^{\eta} + i_{n,2}^{\eta} + 2i_{n,3}^{\eta}) \\ p_{t,n,3} &= p_{t,n,4} = i_{n,3}^{\eta}/(i_{n,1}^{\eta} + i_{n,2}^{\eta} + 2i_{n,3}^{\eta}) \end{aligned} \tag{5}$$

with

$$i_{n,1}=(\sigma_{t,n}+1)(\sigma_{t,n}^2+2\sigma_{t,n})/[(\sigma_{t,n}+2)(\sigma_{t,n}^2+3\sigma_{t,n}+1)]$$
$$i_{n,2}=(\sigma_{t,n}^2+2\sigma_{t,n})/[(\sigma_{t,n}+2)(\sigma_{t,n}^2+3\sigma_{t,n}+1)]$$
$$i_{n,3}=i_{n,4}=\sigma_{t,n}/(3\sigma_{t,n}+1)$$

where $p_{t,n,i}$ is the growth probability on the bond i within the generator for the tip at the nth generation. The growth probability $p_{f,n,i}$ on the bond i within the generator for the fjord at the nth generation is given by

$$p_{f,n,1}=p_{f,n,2}=i_{n,1}^{\eta}/(2i_{n,1}^{\eta}+2i_{n,3}^{\eta})$$
$$p_{f,n,3}=p_{f,n,4}=i_{n,3}^{\eta}/(2i_{n,1}^{\eta}+2i_{n,3}^{\eta}) \tag{6}$$

with

$$i_{n,1}=i_{n,2}=\sigma_{f,n} \qquad i_{n,3}=i_{n,4}=\sigma_{f,n}/(3\sigma_{f,n}+1).$$

The growth probabilities $p_{t,0}$ and $p_{f,0}$ within the initiator are given by

$$p_{t,0}=i_{0,t}^{\eta}/(i_{0,t}^{\eta}+i_{0,f}^{\eta}) \qquad p_{f,0}=i_{0,f}^{\eta}/(i_{0,t}^{\eta}+i_{0,f}^{\eta}) \tag{7}$$

with

$$i_{0,t}=\sigma_{t,0} \qquad i_{0,f}=\sigma_{f,0}/(\sigma_{f,0}+1).$$

The partition function is given by the multiplicative process of the generator's growth probabilities

$$\sum_i p_i^q=p_{f,0}^q(2p_{f,1,1}^q+2p_{f,1,3}^q)(2p_{f,2,1}^q+2p_{f,2,3}^q)\ldots(2p_{f,n,1}^q+2p_{f,n,3}^q)\ldots(2p_{f,N,1}^q+2p_{f,N,3}^q)$$
$$+p_{t,0}^q(p_{t,1,1}^q+p_{t,1,2}^q+2p_{t,1,3}^q)(p_{t,2,1}^q+p_{t,2,2}^q+2p_{t,2,3}^q)$$
$$\ldots(p_{t,n,1}^q+p_{t,n,2}^q+2p_{t,n,3}^q)\ldots(p_{t,N,1}^q+p_{t,N,2}^q+2p_{t,N,3}^q). \tag{8}$$

Here, the system size is given by $L=4^N$.

For sufficiently large N,

$$\lim_{n\to 0}\sigma_{f,n}\to\infty \tag{9}$$

$$\lim_{n\to 0}\sigma_{t,n}\to\sigma_t^* \qquad \text{(finite value)} \tag{10}$$

where σ_t^* is the fixed point of the recursion relation (3).

The growth probabilities within the generators approach to the limiting values

$$\lim_{n\to 0} p_{t,n,1}=p_{t,1}^*(=\text{constant value})$$

$$\lim_{n\to 0} p_{t,n,2}=p_{t,2}^*(=\text{constant value})$$

$$\lim_{n\to 0} p_{t,n,3}(=p_{t,n,4})=p_{t,3}^*(=p_{t,4}^*)(=\text{constant value}) \tag{11}$$

$$\lim_{n\to 0} p_{f,n,1}(=p_{f,n,2})=p_{f,1}^*(=p_{f,2}^*)=1/2$$

$$\lim_{n\to 0} p_{f,n,3}(=p_{f,n,4})\to 0.$$

The maximum growth probability p_{max} scales as

$$p_{max}\approx L^{\ln p_{t,1}^*/\ln 4}. \tag{12}$$

We shall consider the dependence of the minimum growth probability $p_{\min}$ upon the system size L. The minimum growth probability $p_{\min}$ is given by the multiplicative process of the minimum value of the generator's growth probability

$$p_{\min} \approx p_{f,1,3} p_{f,2,3} \cdots p_{f,n,3} \cdots p_{f,N,3} \tag{13}$$

with

$$p_{f,n,3} = (1/3)^{\eta} / [2(\sigma_{f,n})^{\eta} + 2(1/3)^{\eta}]$$

$$\sigma_{f,n-1} = 2\sigma_{f,n} + 2/3 \qquad \text{for sufficiently large } N.$$

The relationship (13) is approximated by

$$p_{\min} \approx 2^{-N} 3^{-\eta N} 2^{-\eta(1+2+\ldots+N)} \{(5/3)^{\eta} + (1/6)^{\eta}\}^{-N}. \tag{14}$$

We obtain (2) for the minimum growth probability. For $q < 0$, the partition function is undefined in the limit of $N \to \infty$.

For $q > 0$

$$(p_{t,1}^{*q} + p_{t,2}^{*q} + 2p_{t,3}^{*q}) > 2p_{f,1}^{*q}. \tag{15}$$

Therefore, the partition function for $q > 0$ is given by

$$\sum_i p_i^q \approx (p_{t,1}^{*q} + p_{t,2}^{*q} + 2p_{t,3}^{*q})^N. \tag{16}$$

Finally, we obtain the following expression of $\tau(q) = (q-1)D(q)$:

$$-(q-1)D(q) = \ln(p_{t,1}^{*q} + p_{t,2}^{*q} + 2p_{t,3}^{*q}) / \ln 4 \qquad (q > 0). \tag{17}$$

We obtain the analytical form of the 'left-sided' multifractality for any η

$$\tau(q) = \begin{cases} \text{undefined} & q > 0 \\ -1 & q = 0 \\ \text{equation (17)} & q > 0. \end{cases} \tag{18}$$

Our result (18) is derived exactly analytically. However, in the Lee *et al* model (1990), the multifractality of the growth probability distribution is obtained from solving numerically the Laplace equation.

Here we give a comment to justify our model. The diamond lattice is not realistic for describing a real lattice, but gives the likely behaviour on the lattice. The multifractality of the growth probability distribution is not affected qualitatively by the lattice type. One can have the advantage of being possible to calculate analytically the growth probability. The self-similar structure of the fjord, which is obtained by the use of the generator for the fjord (figure 1(d)), is similar to the hierarchical model devised by Lee *et al* (1990). The deterministic fractal structure for the fjord in our model has an infinite hierarchy of voids connected by narrow channels. The narrow channels correspond to the bonds 1 and 2 in the generator for the fjord (figure 1(d)). The voids correspond to the bonds on the right-hand side in the generator. The concept of open voids connected by narrow channels has been supported by visual computer simulation of DLA clusters (Lee *et al* 1990). Very recently, Mandelbrot and Evertsz (1990) showed the more qualitative but quite compelling computer simulation results for the fjord structure. They found that points of highest and lowest growth probability can lie unexpectedly close together, and that the lowest growth probabilities may lie very far from the initial seed. In our model, points of highest and lowest growth probability lie in the farthest position and the lowest growth probability lies on the initial seed. The fjord structure in our model is not consistent with that of Mandelbrot and Evertsz (1990). It will be necessary to improve our model to take into account the finding by Mandelbrot and Evertsz (1990).

In summary, we present a deterministic fractal model to mimic a typical DLA cluster. We show that the left-sided multifractality of the growth probability distribution can be derived by solving the resistor network for the model exactly. We find that the model predicts the tip behaviour (the growth probability can scale for $q > 0$) and the fjord behaviour (the minimum growth probability has a logarithmic singularity).

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